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Dynamically SUSY breaking SQCD on F-theory seven-branes

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Abstract: We study how dynamically breaking SQCD can be obtained on two intersecting seven-branes in F-theory. In the mechanism which we present in this paper one of the seven-branes is responsible for producing the low-energy gauge group and the other one is for generating vector bundle moduli. The fundamental matter charged under the gauge group is localized on the intersection. The mass of the matter fields is controlled by the vector bundle moduli. The analysis of under what conditions a sufficient number of the fundamental flavors becomes light turns out to be equivalent to the analysis of non-perturbative superpotentials for vector bundle moduli in Heterotic M-theory. We give an example in which we present an explicit equation in the moduli space whose zero locus corresponds to the fundamental fields becoming light. This allows us to provide a local F-theory realization of massive $\mathcal{N} = 1$, SU(N_c) SQCD in the free magnetic range which dynamically breaks supersymmetry.

Keywords: F-Theory, Intersecting branes models, Supersymmetry and Duality.

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Contents

1. Introduction

F-theory [1] compactified to four dimensions is potentially one of the most promising ways to obtain phenomenological models. One of the attractive features of F-theory is that, unlike most of the intersecting brane models, it naturally provides Grand Unification at the compactification scale. In addition, F-theory compactifications admit a rich structure of branes and fluxes which suggests a potential variety of possibilities to build quasi-realistic phenomenology.

One of the reasons to expect interesting particle physics in F-theory is due to its relation with heterotic string. For a certain type of heterotic compactifications, namely on elliptically fibered Calabi-Yau manifolds, the heterotic/F-theory duality is relatively

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well understood [2 − 10]. Since heterotic compactifications are known to naturally lead to quasi-realistic phenomenological models one should expect the same on the F-theory side. Recently, vacua with the spectrum of the supersymmetric standard model were obtained in heterotic compactifications on non-simply connected Calabi-Yau manifolds [11 − 16]. The relation between heterotic string and F-theory suggests that such models can probably also be found on the F-theory side. On the other hand, the general structure of F-theory compactifications has certain advantages comparing to that in heterotic string theory. The particle sector in F-theory is localized on (in general intersecting) seven-branes. It implies that in order to study particle physics in F-theory one is likely to need to know only the local structure of the F-theory Calabi-Yau manifold near the seven-branes. On the contrary, on the heterotic side, there are no branes involved in the model building and it is not possible to reduce the problem to a local consideration. Another attractive feature of F-theory, or type IIB compactifications in general, is a recent progress in moduli srabilization (see [17, 18] for a review) and cosmological applications (see [19, 20] for the most recent reports).

On the other hand, the particle spectrum of the F-theory compactifications is very poorly understood and its study represents a difficult problem. It is hard to approach this problem from the type IIB string theory side because F-theory compactifications are intrinsically non-perturbative. Away from certain orientifold limits one cannot obtain the spectrum in a simple way by quantizing open strings ending on the F-theory seven-branes. The approach based on using duality with heterotic string theory is also problematic since it is not known how the duality map acts on the heterotic spectrum. In general, it is a complicated mathematical problem. A progress in this direction has recently been reported in [21, 22] (see also [23]). In particular, in [22], Beasley, Heckman and Vafa constructed a field theory on intersecting seven-branes. The approach of [22] was to start with the maximally supersymmetric Yang-Mills theory and twist it in such a way that the theory on the branes preserves only four supercharges. The authors of [22] showed that such a twist is unique. Once the theory on the seven-branes is known one can study the particle spectrum in four dimensions just like in heterotic compactifications. The analysis in [22] relies only on the local geometry near the seven-branes. However, one can expect that it rather adequately describes the particle sector of F-theory compactifications though global restrictions in some cases can be important.

The goal of [22, 24] is to study GUT theories in the F-theory framework. However, in general, it is interesting to look not only at quasi-realistic theories in the visible sector but also at hidden sectors. One of the important questions in string theory model building is how supersymmetry can be broken in these models. A natural attempt would be to create a hidden sector which breaks supersymmetry and then to communicate this breaking to the visible sector via some mediation mechanism. The most recent progress on dynamical SUSY breaking was achieved by Intrilligator, Seiberg and Shih in [25] where it was shown that a class of $\mathcal{N} = 1$ SQCD theories has a metastable SUSY breaking vacuum at strong coupling. This class involves theories whose matter spectrum consists of N_f massive fundamental flavors where N_f is in the free magnetic range. It is important to understand how field theories dynamically breaking SUSY in the infrared, like the one studied in [25], can be embedded in realistic string compactifications with stable moduli. This has been discussed in various contexts in [26 − 28].

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In this paper, we will discuss how the field theory model of [25], that is massive SQCD in the free magnetic range can be obtained on the seven-branes of F-theory. More precisely, we consider the theory on two intersecting seven-branes. The field theory action for this system was obtained in [22]. Since we have a four-dimensional theory, the seven-branes wrap two different complex surfaces and extend in four dimensions. In addition, they intersect along a complex curve. All these objects, namely the two seven-branes and the curve play an important role in our construction. The theory on one of the seven-branes is pure $\mathcal{N}=1$, $\mathrm{SU}(N_c)$ Yang-Mills theory without matter. The role of the second seven-brane is to contribute vector bundle moduli. To achieve it, we put a non-trivial instanton on the surface which this seven-brane wraps. The matter in the (anti)-fundamental representation of the gauge group $\mathrm{SU}(N_c)$ comes from the intersection curve. In order to generate the field theory of [25] the matter has to receive a relatively small mass. In global compactifications, there are no free constant parameters which can be used for this purpose. The role of parameters is played by moduli fields which have to be stabilized in any quasi-realistic compactification. In our case, the relevant moduli are the moduli of the vector bundle. The mass of the fundamental fields localized on the intersection curve is a function of these moduli. In fact, in a generic point in the moduli space, all the matter fields are very massive and have to be integrated out at low energies. The resulting theory in this case is $\mathrm{SU}(N_c)$ supersymmteric Yang-Mills theory. However, near some special subvarieties in the moduli space a certain number of the fundamental fields can become light and the resulting theory is SQCD with slightly massive flavors. If one can control how many fundamentals become light near various subvarieties in the moduli space, one can obtain massive SQCD in the free magnetic range. We show that this problem of analyzing under what conditions there is light fundamental matter is exactly equivalent to the problem of computing non-perturbative superpotentials for vector bundle moduli in Heterotic M-theory [29 − 32]. The holomorphic function which was the superpotential in the Heterotic M-theory context now defines the subvariety near which there are light fundamental fields. To have an analytic control over the problem, we choose one of the surfaces to be the rational elliptic surface dP_9. This surface admits an elliptic fibration so that we can use the spectral cover construction [5, 33] to build an instanton on it. As the result, we can write an explicit equation in the moduli space which governs the appearance of light fundamental fields as well as their number. More precisely, the fundamental flavors parametrize the kernel of a certain square matrix. Therefore, the number light flavors coincides with the amount by which the rank of this matrix drops as we move in the locus of the zero determinant.

This paper is organized as follows. In section 2, we give a review of F-theory compactifications. In particular, we review the field theory on intersecting seven-branes constructed in [22] with focus on how the four-dimensional particle spectrum is encoded in the geometry of the branes. In subsection 3.1, we state, following [25], the criteria that field theories with dynamical supersymmetry breaking must satisfy. In the rest of section 3, we study how such theories can be embedded in F-theory. In section 4, we present a concrete realization of the ideas developed in section 3. We give an example where the holomorphic function in the moduli space, near the zero locus of which we obtain massless fundamental matter, can

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be explicitly derived. We analyze how many fundamental flavors become light in different regimes in the moduli space and give examples of SQCD in the free magnetic range. In addition, we show that the mechanism of generating light fields as we move in the vector bundle moduli space is precisely equivalent to having a Yukawa-type superpotential in the Lagrangian. This superpotential is quadratic in the matter fields with the mass matrix depending on the vector bundle moduli. In conclusion, we briefly summarize our results and discuss a possible extension of this work. Finally, appendices A, B and C are devoted to some technical details.

2. F-theory compactifications

2.1 The general structure

In this section, we will review the structure of F-theory compactifications [1 − 6]. We will start with a general consideration and then review details of the field theory on the seven-branes [22].

F-theory is a special class of supersymmetric type IIB string compactifications on a manifold which we will denote Y. This compactification has a non-trivial holomorphic axion-dilation which varies along Y. It can become singular and undergo an $\mathrm{SL}(2,\mathbb{Z})$ monodromy along some divisor in Y which we will denote Δ. This can be interpreted as a compactification on a Calabi-Yau manifold X which is elliptically fibered over Y with Δ being the discriminant divisor of the the elliptic fibration over which the fibers degenerate. The position of Δ is interpreted as the location of the seven-branes on which the particle sector of the compactification is localized. The gauge group is determined by the type of the singularity along Δ. In this paper, we will be interested in compactifications to four dimensions. Then X is a Calabi-Yau four-fold, elliptically fibered over Y, and Δ is a surface in Y. In many cases, Δ is reducible and has irreducible components intersection along a curve or points. In such situations, the particle sector can be viewed as an intersecting brane model where the seven-branes wrap surfaces in the four-fold X and extend in the four non-compact dimensions.

To provide a global realization of such Calabi-Yau four-folds is a rather complicated task. However, there is a class of F-theory compactifications whose global properties are relatively well understood. These are F-theory n-fold compactifications dual to heterotic compactifications on an elliptically fibered Calabi-Yau $(n-1)$-fold with a vector bundle whose structure group is in $E_8 \times E_8$.[1] Let us give a brief review of the Calabi-Yau manifold X is this case. This will provide us with some intuition about the general structure of the F-theory compactifications which will be used to motivate some of the choices we make further in the paper. For concreteness, we will discuss the case $n = 4$ which corresponds to compactifications to four dimensions on both sides of the duality. In this case, the F-theory four-fold is described by a Weierstrass model

$$y^2 = x^3 + f(z_0; z_1, z_2)x + g(z_0; z_1, z_2). \tag{2.1}$$

[1]There is additional data involved in this duality. The vector bundle on the heterotic side has to be constructed using an *irreducible* spectral cover. For simplicity, we will omit these details.

For fixed $(z_0; z_1, z_2)$ this equation describes an elliptic fiber. The coordinate z_0 parametrizes $\mathbb{P}^1$ and $f(z_0; z_1, z_2)$ and $g(z_0; z_1, z_2)$ are polynomials of degree eight and twelve in z_0 respectively

$$
\begin{aligned}
f(z_0; z_1, z_2) &= \sum_{a=0}^{8} f_a(z_1, z_2) z_0^a, \\
g(z_0; z_1, z_2) &= \sum_{b=0}^{12} f_b(z_1, z_2) z_0^b.
\end{aligned}
\tag{2.2}
$$

For fixed (z_1, z_2) eqs. (2.1), (2.2) describe elliptically fibered $K3$ surface. Indeed, the discriminant of the fibration given by

$$
\Delta = 4f^3 + 27g^2 \tag{2.3}
$$

is a polynomial of degree 24 in z_0 meaning that the fiber degenerates over 24 point is $\mathbb{P}^1$. Thus, X is also a $K3$ fibration over a complex two-dimensional manifold parametrized by (z_1, z_2). This space is identified with the base B of the elliptically fibered three-fold on the heterotic side. In other words, elliptically fibered Calabi-Yau threefold with base B on the heterotic side is mapped by duality to the F-theory elliptically fibered Calabi-Yau four-fold X given by a Weierstrass model (2.1) which is also $K3$ fibered over B. It was shown that the middle coefficients $f_4(z_1, z_2)$ and $g_6(z_1, z_2)$ in eqs. (2.2) encode the information about the complex structure of the heterotic threefold. The coefficients $f_a, a = 0, \ldots, 3$ and g_b, $b = 0, \ldots, 6$ encode the information about one of the E_8 vector bundles. Similarly, the coefficients f_a, $a = 5, \ldots, 8$ and g_b, $b = 7, \ldots, 12$ describe the data of the other E_8 vector bundle. Thus, to describe one of the particle sectors (visible or hidden) one can set the data of the second vector bundle to zero. Then $f(z_0; z_1, z_2)$ can be taken to be a polynomial of degree four in z_0 and $g(z_0; z_1, z_2)$ can be taken to be a polynomial of degree six.

Let us now review the structure of the seven-branes. It is determined by the equation $\Delta = 0$. The low-energy gauge group is determined by the singularity along the zero locus of Δ and is of the ADE-type. All possible consistent singularities were obtained in [4] using the Tate's algorithm. In most cases, the discriminant divisor consists of two components intersecting along the curve $z_0 = 0$ which is just the base of the $K3$-fibration B. For example, the E_7 low-energy gauge group is described by the following coefficients $f(z_0; z_1, z_2)$ and $g(z_0; z_1, z_2)$

$$
\begin{aligned}
f(z_0; z_1, z_2) &= f_4(z_1, z_2) z_0^4 + f_3(z_1, z_2) z_0^3, \\
g(z_0; z_1, z_2) &= g_6(z_1, z_2) z_0^6 + g_5(z_1, z_2) z_0^5.
\end{aligned}
\tag{2.4}
$$

The discriminant divisor can be obtained from eq. (2.3) and is given by the zero locus of the following polynomial

$$
\Delta = z_0^9 (4f_3^3 + 12 f_3^2 f_4 z_0 + (27 g_5^2 + 54 g_5 g_6) z_0^2 + (4 f_4^3 + 27 g_6^2) z_0^3). \tag{2.5}
$$

We see that the discriminant divisor has two components. One is given by $z_0 = 0$ and the other one given by

$$
4f_3^3 + 12 f_3^2 f_4 z_0 + (27 g_5^2 + 54 g_5 g_6) z_0^2 + (4 f_4^3 + 27 g_6^2) z_0^3 = 0. \tag{2.6}
$$

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These two surfaces intersect along the curve $f_3(z_1, z_2) = 0$. This and all other possible ADE-singularities were studied in detail in [4].

2.2 The theory on the intersecting seven-branes

To describe the particle spectrum of F-theory compactifications one has to study the theory on the seven-branes. This analysis was performed in [22]. Motivated by the structure of the seven-branes in the globally known examples of F-theory reviewed in the previous subsection, we will concentrate on the models in which the discriminant divisor consists of two smooth irreducible surfaces $\mathcal{S}$ and $\mathcal{S}'$ intersecting along a curve Σ which, for simplicity, we will assume to be smooth and irreducible. Let the singularities along $\mathcal{S}$ and $\mathcal{S}'$ be of the type $G_{\mathcal{S}}$ and $G_{\mathcal{S}'}$ respectively. Both $G_{\mathcal{S}}$ and $G_{\mathcal{S}'}$ are of the ADE-type. This corresponds to $G_{\mathcal{S}}$ and $G_{\mathcal{S}'}$ gauge groups on the two intersecting seven-branes. We should note that one can also have a situation when there is no singularity along $\mathcal{S}$ or $\mathcal{S}'$. In this case, the gauge group on the world-volume of the corresponding seven-brane is $\mathrm{U}(1)$.

To describe the theory on a seven-brane, the authors of [22] started with the maximally supersymmetric gauge theory on $\mathbb{R}^{1,3} \times \mathbb{C}^2$. Then they replaced $\mathbb{C}^2$ with the component of the discriminant surface $\mathcal{S}$. The theory on $\mathbb{R}^{1,3} \times \mathcal{S}$ has to preserve four supercharges. It was shown in [22] that this can achieved if one twists the maximally supersymmetric theory on $\mathbb{R}^{1,3} \times \mathbb{C}^2$. Furthermore, it was shown in [22] that there exists a unique twists preserving four supercharges. In a similar manner, one can analyze the theory on $\mathbb{R}^{1,3} \times \Sigma$ [22]. To make the paper self-contained, we give a review of the twisting in appendices A and B.

The resulting action of the theory on the intersecting seven-branes is

$$I = I_{\mathcal{S}} + I_{\mathcal{S}'} + I_{\Sigma}. \tag{2.7}$$

Here $I_{\mathcal{S}}$ is the action localized on $\mathbb{R}^{1,3} \times \mathcal{S}$, $I_{\mathcal{S}'}$ is the action localized on $\mathbb{R}^{1,3} \times \mathcal{S}'$ and I_{Σ} is the action localized on the intersection $\mathbb{R}^{1,3} \times \Sigma$. The precise form of $I_{\mathcal{S}}, I_{\mathcal{S}'}$ and I_{Σ} can be found in [22]. Here, we will only review the field content.

Let us start with the fields localized on $\mathbb{R}^{1,3} \times \mathcal{S}$. The first set of fields is

$$(A_\mu, \eta_\alpha, \bar{\eta}_{\dot{\alpha}}), \quad \mu = 0, \dots 3, \quad \alpha = 1, 2 \tag{2.8}$$

which can be viewed as the vector multiplet. Here A_μ is the four-dimensional part of the $G_{\mathcal{S}}$-gauge field propagating on $\mathbb{R}^{1,3} \times \mathcal{S}$. Furthermore, η_α is a positive chirality spinor from the viewpoint of the four-dimensional Lorentz group. It also transforms in the adjoint representation of $G_{\mathcal{S}}$ (more precisely, it is a section of the adjoint bundle on $\mathbb{R}^{1,3} \times \mathcal{S}$). So far, all the fields in (2.8) depend on the coordinates on $\mathbb{R}^{1,3}$ as well on the coordinates on $\mathcal{S}$. The additional field can be viewed as (anti)-chiral multiplets

$$(A_m, \bar{\psi}_{\dot{\alpha} m}), \quad (A_{\bar{m}}, \psi_{\alpha \bar{m}}) \tag{2.9}$$

and

$$(\phi_{mn}, \chi_{\alpha mn}), \quad (\bar{\phi}_{\bar{m}\bar{n}}, \bar{\chi}_{\dot{\alpha}\bar{m}\bar{n}}), \tag{2.10}$$

where $m, n = 1, 2$, is the holomorphic index on $\mathcal{S}$. The fermions $\bar{\psi}_{\dot{\alpha} m}$ and $\chi_{\alpha mn}$, in addition to being sections of the adjoint bundle, transform as sections of $\Omega^{(1,0)}_{\bar{\partial}}$ and $\Omega^{(2,0)}_{\bar{\partial}}$

respectively. All field in eqs. (2.9) and (2.10) depend on the coordinates on $\mathbb{R}^{1,3}$ as well on the coordinates on $\mathcal{S}$. To obtain the low-energy field theory in four-dimensions we compactify the action $I_{\mathcal{S}}$ on $\mathcal{S}$ and keep only the zero modes. To preserve supersymmetry, we have to satisfy some BPS conditions on $\mathcal{S}$. These conditions are as follows [22]

$$\begin{aligned} F_{mn} = F_{\bar{m}\bar{n}} &= 0, \\ \bar{\partial}_{A_m}\phi &= 0, \\ \omega \wedge F + \frac{i}{2}[\phi, \bar{\phi}] &= 0. \end{aligned} \tag{2.11}$$

Here F is the field strength constructed out of $(A_m, A_{\bar{m}})$. It can be viewed as a curvature of some vector bundle on $\mathcal{S}$. Furthermore, $\phi = \phi_{mn} ds^m ds^n$ is an adjoint-valued two-form on $\mathcal{S}$, $\bar{\partial}_{A_m}$ is the antiholomorphic covariant derivative and ω is the Kahler form. For simplicity, we will consider vacua with $\phi = 0$. Then the equations for F become

$$F_{mn} = 0, \quad F_{\bar{m}\bar{n}} = 0, \quad g^{m\bar{n}} F_{m\bar{n}} = 0 \tag{2.12}$$

which are the Hermitian Yang-Mills equations on $\mathcal{S}$. This means that F is the curvature on a stable holomorphic vector bundle on $\mathcal{S}$. Let $H_{\mathcal{S}}$ be the structure group of this vector bundle. Then in four dimensions $G_{\mathcal{S}}$ is broken to $\Gamma_{\mathcal{S}}$ which is the commutant of $H_{\mathcal{S}}$ in $G_{\mathcal{S}}$. Thus, after compactifying on $\mathcal{S}$ the action $I_{\mathcal{S}}$ is the action of the $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $\Gamma_{\mathcal{S}}$ coupled to some matter. To obtain the matter content, we first decompose $\mathrm{ad}G_{\mathcal{S}}$ into irreducible representations of $\Gamma_{\mathcal{S}} \times H_{\mathcal{S}}$

$$\mathrm{ad}G_{\mathcal{S}} = \bigoplus_j \tau_j \otimes \mathcal{T}_j. \tag{2.13}$$

Since the light fermionic matter is given by the zero modes of the Dirac operator on $\mathcal{S}$ it follows that the fermionic spectrum is given by

$$\begin{aligned} \bar{\eta}_{\dot{\alpha},\tau_j} &\in H^0(\mathcal{S}, T_j), \\ \psi_{\alpha,\tau_j} &\in H^1(\mathcal{S}, T_j), \\ \bar{\chi}_{\dot{\alpha},\tau_j} &\in H^2(\mathcal{S}, T_j), \end{aligned} \tag{2.14}$$

where T_j is the vector bundle on $\mathcal{S}$ whose sections transform in the representation $\mathcal{T}_j$ of the structure group $H_{\mathcal{S}}$. The upper index in the cohomology groups H^i is due to the fact that the fermions are twisted. Of course, the spectrum in eq. (2.14) has to be supplemented by the complex conjugate fields. Note that the term in eq. (2.13) corresponding to $\tau_j = \mathbf{1}$, $\mathcal{T}_j = \mathrm{ad}H_{\mathcal{S}}$ counts the vector bundle moduli. As the result, the chiral spectrum is given by [22]

$$H^0(\mathcal{S}, T_j^\vee)^\vee \oplus H^1(\mathcal{S}, T_j) \oplus H^2(\mathcal{S}, T_j^\vee)^\vee \tag{2.15}$$

and the antichiral spectrum is given by [22]

$$H^0(\mathcal{S}, T_j) \oplus H^1(\mathcal{S}, T_j^\vee)^\vee \oplus H^2(\mathcal{S}, T_j), \tag{2.16}$$

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where the symbol $\vee$ stands for the dual bundle or vector space. The difference between the chiral and antichiral matter in the representation τ_j of $\Gamma_{\mathcal{S}}$ is given by the difference of the Euler characteristics

$$n_{\tau_j} - n_{\tau_j^*} = \chi(\mathcal{S}, \mathcal{T}_j^\vee) - \chi(\mathcal{S}, \mathcal{T}_j) = -\int c_1(\mathcal{T}_j)c_1(\mathcal{S}), \tag{2.17}$$

where $c_1(\mathcal{T}_j)$ and $c_1(\mathcal{S})$ are the first Chern classes of $\mathcal{T}_j$ and the holomorphic tangent bundle of $\mathcal{S}$ respectively.

The analysis of the theory of the seven-branes wrapping $\mathcal{S}'$ is identical to the one presented above.

Let us now discuss the theory localized on $\mathbb{R}^{1,3} \times \Sigma$. It was also obtained in [22] by twisting the maximally supersymmetric gauge theory in six dimensions. The result is that on the intersection one gets two chiral multiplets

$$(\sigma, \lambda_\alpha), \quad (\sigma^c, \lambda_\alpha^c), \tag{2.18}$$

where all these fields transform in representation of $G_{\mathcal{S}} \times G_{\mathcal{S}'}$. An additional important feature is that the fields in (2.18) are twisted and transform as sections of $K_\Sigma^{1/2}$, where K_Σ is the canonical bundle on Σ. Note that since Σ is a Riemann surface it is a spin manifold and the square root of the canonical bundle exists. To obtain which representations of $G_{\mathcal{S}} \times G_{\mathcal{S}'}$ are allowed one needs to know how the singularity is enhanced along Σ. Let the singularity be enhanced to another ADE-type group $G_\Sigma \supset G_{\mathcal{S}} \times G_{\mathcal{S}'}$. To obtain the allowed representations of $G_{\mathcal{S}} \times G_{\mathcal{S}'}$ we decompose $\mathrm{ad}G_\Sigma$ as

$$\mathrm{ad}G_\Sigma = \mathrm{ad}G_{\mathcal{S}} \oplus \mathrm{ad}G_{\mathcal{S}'} \oplus \bigoplus_j (\mathcal{U}_j \otimes \mathcal{U}_j'). \tag{2.19}$$

The bifundamentals (σ, λ_α), $(\sigma^c, \lambda_\alpha^c)$ transform in the representations of $G_{\mathcal{S}} \times G_{\mathcal{S}'}$ given by the non-adjoint summand $\bigoplus_j (\mathcal{U}_j \otimes \mathcal{U}_j')$ in (2.19). To obtain the low-energy field theory in four dimension we compactify I_Σ on Σ. As was discussed above, we can put non-trivial instantons on both $\mathcal{S}$ and $\mathcal{S}'$ with structure groups $H_{\mathcal{S}}$ and $H_{\mathcal{S}'}$ respectively. Then the matter fields originating from the intersection multiplets (2.18) will transform in representations of $\Gamma_{\mathcal{S}} \times \Gamma_{\mathcal{S}'}$, where $\Gamma_{\mathcal{S}}$ is the commutant of $H_{\mathcal{S}}$ in $G_{\mathcal{S}}$ and $\Gamma_{\mathcal{S}'}$ is the commutant of $H_{\mathcal{S}'}$ in $G_{\mathcal{S}'}$. More precisely, we decompose

$$\mathcal{U} \otimes \mathcal{U}' = \bigoplus_k (\nu_k, \mathcal{V}_k), \tag{2.20}$$

where ν_k is a representation of $\Gamma = \Gamma_{\mathcal{S}} \times \Gamma_{\mathcal{S}'}$ and $\mathcal{V}_k$ is a representation of $H = H_{\mathcal{S}} \times H_{\mathcal{S}'}$. The chiral fermions in the representation ν_k correspond to the zero modes of the Dirac operator on Σ. Thus,

$$\lambda_{\alpha,\nu_k} \in H^0(\Sigma, K_\Sigma^{1/2} \otimes V_k), \tag{2.21}$$

$$\lambda_{\alpha,\nu_k}^c \in H^0(\Sigma, K_\Sigma^{1/2} \otimes V_k^\vee) \simeq H^1(\Sigma, K_\Sigma^{1/2} \otimes V_k)^\vee, \tag{2.22}$$

where in the last step in (2.22) we have used the Serre duality on Σ and V_k is the vector bundle whose sections transform in the representation $\mathcal{V}_k$ of H. The additional factor $K_\Sigma^{1/2}$

in eqs. (2.21) and (2.22) is due to the fact that the fermions are twisted. The difference between chiral and antichiral matter in the representation ν_k of the low-energy gauge group Γ is given by the Euler characteristic

$$n_{\nu_k} - n_{\nu_k^*} = \chi(\Sigma, K_\Sigma^{1/2} \otimes V_k). \tag{2.23}$$

This concludes our review of the theory on the intersecting seven-branes. Additional details can be found in [22].

3. Dynamical SUSY breaking

3.1 Field theory requirements

In this subsection, we will give a brief review of dynamical supersymmtery breaking in $\mathcal{N} = 1$ SQCD following [25]. The goal is to formulate the field theory requirements which we will intend to realize on F-theory seven-branes. We consider $\mathcal{N} = 1, \mathrm{SU}(N_c)$ SQCD with N_f fundamental flavors $Q, \tilde{Q}$ in the free magnetic range [34, 35]

$$N_c + 1 \leq N_f < \frac{3}{2} N_c. \tag{3.1}$$

The flavors are taken to be massive and have a quadratic superpotential

$$W = \mathrm{Tr} m M, \tag{3.2}$$

where

$$M = Q_f \cdot \tilde{Q}_g, \quad f, q = 1, \ldots N_f. \tag{3.3}$$

This theory is known to have N_c supersymmetric vacua with

$$\langle M \rangle = \left(\Lambda^{3N_c - N_f} \det m\right)^{1/N_c} m^{-1}, \tag{3.4}$$

where Λ is the strong coupling scale. It was shown in [25] that, in addition, this theory has a metastable SUSY breaking vacuum. This was established by studying the Seiberg dual [34, 35] of the original theory. The Seiberg dual theory is $\mathrm{SU}(N_f - N_c)$ SQCD with N_f fundamentals q, $\tilde{q}$ and N_f^2 extra singlets Φ_f^g. It has a quadratic leading order Kahler potential and the superpotential given by (up to some field redefinition)

$$W_{\text{dual}} = h \mathrm{Tr} q \Phi \tilde{q} - h \mu^2 \mathrm{Tr} \Phi, \tag{3.5}$$

where $\mu = \sqrt{m\Lambda}$ and h is a dimensionless parameter (see [25] for additional details). For simplicity, we have assumed that all eigenvalues of the mass matrix are equal. This theory breaks supersymmetry by a rank condition mechanism since F-flatness for Φ requires that

$$\tilde{q}^f q_g = \mu^2 \delta_g^f, \tag{3.6}$$

which cannot be satisfied because the number of colors of the dual theory $N_f - N_c$ is less than the number of flavors N_f. However, it was shown in [25] that there exists a metastable SUSY breaking vacuum with

$$V_{\min} = N_c |h^2 \mu^4|. \tag{3.7}$$

This result can be well-trusted in the regime

$$\epsilon \sim \sqrt{\left|\frac{m}{\Lambda}\right|} \ll 1, \tag{3.8}$$

These results were also generalized in [25] for SQCD with gauge groups SO(N_c) and Sp(N_c). In this paper, we will concentrate on the SU(N_c) theories.

3.2 Embedding in F-theory compactifications

In the rest of the section, we will discuss how the field theory reviewed above can be obtained on the intersecting seven-branes in F-theory. We would like to build a massive SU(N_c) SQCD with fundamental matter so that the requirement (3.1) is satisfied. In addition, the fundamental fields have to be very light. As we discussed in the previous section, the charged matter is in one-to-one correspondence with various bundle cohomology groups. The dimensions of bundle cohomology groups are not topological invariants. Thus, they depend on the location in the vector bundle and complex structure moduli spaces. As we move in the moduli space the dimensions might jump meaning that some extra charged matter fields might become light. Physically, this means that a certain number of matter fields have a quadratic superpotential with the mass matrix depending on the moduli of the vector bundle on the complex structure of the F-theory four-fold X. Somewhere in the moduli space, some eigenvalues of the mass matrix can vanish increasing the number of the massless fields. If the difference between the chiral and anti-chiral fermions in some representation of the low-energy gauge group Γ is non-zero then a certain number of matter fields will always stay massless since they are protected by the topological invariants (2.17) or (2.23). Hence, to build SQCD, we are interested in F-theory models with vanishing topological invariants (2.17) and (2.23). Furthermore, we are interested in models where at a generic point in the moduli space all matter is massive. However, near some subvarieties the mass of some fields has to be become light which is also a requirement to generate the field theory from the previous subsection. Note that moduli are dynamical fields in the four-dimensional low-energy fields theory. However, eventually, we are interested in compactifications in which all the modul are stabilized. Thus, we will assume that it is indeed the case and will view them as parameters. We will not discuss the issues of moduli stabilization in this paper.

Let us point out that from a conceptual viewpoint engineering of SQCD with massive flavors in the context of F-theory is not much different from that in the case of flat non-compact branes studied in [36 − 41]. In both cases one has to take a certain number of intersecting branes and by using open string moduli engineer a mass term for the charged matter fields. In the context of [36 − 41] the corresponding open string moduli are brane separations and in the context of F-theory the open string moduli are the vector bundle moduli. However, from a technical viewpoint our case is, clearly, more complicated. Since we are interested in quasi-realistic particle physics compactifications, the seven-branes have to wrap non-trivial compact four-cycles. In addition, these four-cycles are endowed with a non-trivial vector bundle. Since the mass term for the fundamental fields is controlled by the vector bundle moduli to understand the structure of the quadratic superpotential one has to take into a account the geometric properties of the four-cycle and of the vector bundle.

Below we will discuss a class of F-theory models which can lead to SQCD on the intersecting seven-branes with requirements formulated in the previous subsection. In the rest of the section, we will give a general consideration. In the next section we will apply the ideas developed in this section for a concrete example of geometry of the seven-branes.

3.3 The spectrum localized on the surfaces

First, we will consider the sector of the theory that comes from the surfaces $\mathcal{S}$ and $\mathcal{S}'$. Let V be an instanton on $\mathcal{S}$ with structure group $H_{\mathcal{S}}$ and V' be an instanton on $\mathcal{S}'$ with structure group $H_{\mathcal{S}'}$. If $G_{\mathcal{S}}$ and $G_{\mathcal{S}'}$ are the singularities along $\mathcal{S}$ and $\mathcal{S}'$, the low-energy gauge group is $\Gamma_{\mathcal{S}} \times \Gamma_{\mathcal{S}'}$ with $\Gamma_{\mathcal{S}}(\Gamma_{\mathcal{S}'})$ being the commutant of $H_{\mathcal{S}}(H_{\mathcal{S}'})$ in $G_{\mathcal{S}}(G_{\mathcal{S}'})$. At this point let us simplify our model. For concreteness, let us choose the singularity along both $\mathcal{S}$ and $\mathcal{S}'$ to be of the A-type. We will assume that one of the factors in $\Gamma_{\mathcal{S}} \times \Gamma_{\mathcal{S}'}$, say $\Gamma_{\mathcal{S}}$, is trivial as far as the gauge theory dynamics is concerned. There are several natural ways to achieve it. The simplest way is to put an instanton on $\mathcal{S}$ with structure group $G_{\mathcal{S}}$. This way we find that $\Gamma_{\mathcal{S}}$ is completely broken. We also do not obtain any massless matter in the $\mathcal{S}$-sector except for the vector bundle moduli. One more way is to take $\Gamma_{\mathcal{S}}$ to be U(1). Since U(1) is infrared free it does not effect the strong coupling dynamics of any non-Abelian factor and, hence, can be ignored. Another way is to assume that the volume of $\mathcal{S}$ is much bigger than the volume of $\mathcal{S}'$. Then the coupling constant of $\Gamma_{\mathcal{S}}$ is parametrically much smaller than the coupling constant of $\Gamma_{\mathcal{S}'}$. In this case, $\Gamma_{\mathcal{S}}$ can be viewed as an (approximate) global symmetry. In this paper, we will concentrate on the first possibility. That is, we will put an instanton on $\mathcal{S}$ with structure group $G_{\mathcal{S}}$ which we will denote SU(n)

$$H_{\mathcal{S}} = G_{\mathcal{S}} = \mathrm{SU}(n). \tag{3.9}$$

In principle, one can keep the $\mathcal{S}'$-sector non-trivial and generate SQCD with the product gauge group which also might break SUSY in the infrared [39, 42 − 44]. However, we will simplify our model and concentrate on the theory of [25] reviewed in the previous subsection.

Furthermore, we will put the trivial instanton on the other surface $\mathcal{S}'$. Then the gauge group $\Gamma_{\mathcal{S}'}$ equals $G_{\mathcal{S}'}$ which we will denote SU(N_c). That is,

$$\Gamma_{\mathcal{S}'} = G_{\mathcal{S}'} = \mathrm{SU}(N_c). \tag{3.10}$$

Let us study the spectrum of the theory. In the $\mathcal{S}'$-sector we obtain $\mathcal{N} = 1$, SU(N_c) supersymmetric gauge theory without any matter. In the $\mathcal{S}$-sector the only fields are the moduli of V, which we view as parameters.

Let us now explain why we have chosen to put the trivial instanton on $\mathcal{S}'$. For this we have to see how the spectrum of the theory in the $\mathcal{S}'$-sector gets modified if the instanton V' on $\mathcal{S}'$ is non-trivial. Let us specify the low-energy gauge group $\Gamma_{\mathcal{S}'}$ to be SU(N_c) as before. Since in the presence of a non-trivial vector bundle on $\mathcal{S}'$ it does not coincide with $G_{\mathcal{S}'}$, we will denote $G_{\mathcal{S}'} = \mathrm{SU}(N)$, $N > N_c$. The structure group of the vector bundle is then $\mathrm{SU}(N-N_c)$. Note that, to be precise, the low-energy gauge group in this case is $\mathrm{SU}(N_c)\times \mathrm{U}(1)$ but as we discussed the $\mathrm{U}(1)$-factor is irrelevant for our purposes and will be ignored. The spectrum of the theory in the $\mathcal{S}'$-sector consists of the $\mathrm{SU}(N_c)$ vector multiplet, the moduli of the vector bundle V' that we put on $\mathcal{S}'$ and the matter fields charged under $\mathrm{SU}(N_c)$. According to our consideration in the previous section, in order to obtain the matter content, we have to decompose the adjoint representation of $\mathrm{SU}(N)$ under $\mathrm{SU}(N_c)\times\mathrm{SU}(N-N_c)$. The fields charged under $\mathrm{SU}(N_c)$ are contained in the following terms of the decomposition (ignoring the $\mathrm{U}(1)$-charges)

$$(\mathbf{N_c},\overline{\mathbf{N}-\mathbf{N_c}})\oplus(\overline{\mathbf{N}}_\mathbf{c},\mathbf{N}-\mathbf{N_c}). \tag{3.11}$$

Thus, the matter charged under $\Gamma_{\mathcal{S}'}=\mathrm{SU}(N_c)$ corresponds to the cohomology groups with coefficients in V' and $V'^{\vee}$. From the previous section it follows that the number of fields in the fundamental representation of $\Gamma_{\mathcal{S}}$ is[2]

$$h^0(\mathcal{S}',V')+h^1(\mathcal{S}',V'^{\vee})+h^2(\mathcal{S}',V') \tag{3.12}$$

whereas the number of the antifundamental fields is

$$h^0(\mathcal{S}',V'^{\vee})+h^1(\mathcal{S}',V')+h^2(\mathcal{S}',V'^{\vee}). \tag{3.13}$$

Since V' is a stable bundle, $h^0(\mathcal{S}',V')=h^0(\mathcal{S}',V'^{\vee})=0$. Furthermore, using the Serre duality we have

$$h^2(\mathcal{S}',V'^{\vee})=h^0(\mathcal{S}',V'\otimes K_{\mathcal{S}'}), \tag{3.14}$$

where $K_{\mathcal{S}'}$ is the canonical bundle on $\mathcal{S}'$. In many cases $V'\otimes K_{\mathcal{S}'}$ also does not have sections and the right hand side in (3.14) vanishes. For instance, this is the case if $\mathcal{S}'$ is one of del Pezzo surfaces [22]. We will assume that $h^2(\mathcal{S}',V')=h^2(\mathcal{S}',V'^{\vee})=0$. Then the matter charged under $\Gamma_{\mathcal{S}'}=\mathrm{SU}(N_c)$ receives a contribution only from $h^1(\mathcal{S}',V')$ and $h^1(\mathcal{S}',V'^{\vee})$. It then follows that it is given by the Euler characteristics

$$h^1(\mathcal{S}',V')=-\chi(\mathcal{S}',V') \tag{3.15}$$

and

$$h^1(\mathcal{S}',V'^{\vee})=-\chi(\mathcal{S}',V'^{\vee}). \tag{3.16}$$

In other words, the number of (anti)-fundamentals is given by topological invariants and protected from becoming massive unless $\chi(\mathcal{S}',V')=\chi(\mathcal{S}',V'^{\vee})=0$. This explains why we did not put a non-trivial instanton on the same seven-brane $\mathcal{S}'$ which carries the $\mathrm{SU}(N_c)$ gauge theory. Putting a non-trivial vector bundle on $\mathcal{S}'$ would generate (anti)-fundamental matter of the gauge group $\mathrm{SU}(N_c)$. This matter would be topologically protected from becoming massive unless $\chi(\mathcal{S}',V')=\chi(\mathcal{S}',V'^{\vee})=0$ which is a strong restriction. Hence, it would be difficult to generate SQCD with massive flavors in this case.

To summarize, the spectrum of the theory localized on the surfaces is pure $\mathrm{SU}(N_c)$ gauge theory with some number of vector bundle moduli. The fundamental matter charged under $\mathrm{SU}(N_c)$ comes from the sector localized on the intersection curve Σ which we have to discuss next.

[2]Throughout the paper we denote by H^i cohomology groups and by h^i their dimension.

3.4 The spectrum localized on the curve

Now let us discuss the theory in the Σ-sector. First, we need to specify the enhanced singularity along Σ. We chose the singularities along $\mathcal{S}$, $\mathcal{S}'$ and Σ to be of the A-type. In notation of the previous subsection, the matter fields on Σ, which we denote as (σ, λ_α) and $(\sigma^c, \lambda^c_\alpha)$, transform as the (anti)-fundamentals of the group $\mathrm{SU}(n) \times \mathrm{SU}(N_c)$. When we compactify to four dimensions, the $\mathrm{SU}(N_c)$ factor survives as the gauge symmetry and the $\mathrm{SU}(n)$ factor is completely broken by the vector bundle. Therefore, the massless four-dimensional fields transform as (anti)-fundamentals of $\mathrm{SU}(N_c)$. Indeed, the non-adjoint summand in eq. (2.19) is our case is

$$(\mathbf{N_c}, \overline{\mathbf{n}}) \oplus (\overline{\mathbf{N}}_\mathbf{c}, \mathbf{n}) \tag{3.17}$$

The fields corresponding to the first terms are (σ, λ_α) and the fields corresponding to the second term are $(\sigma^c, \lambda^c_\alpha)$. When we compactify on Σ, the first entry in both terms in (3.17) labels the representation of the low-energy gauge group $\mathrm{SU}(N_c)$ and the second entry specifies the vector bundle. Recalling that the fields on the intersection are twisted by the square root of the canonical bundle, we obtain the following matter content. We have the multiplets $(\tilde{Q}, \tilde{\lambda}_\alpha)$ whose number is determined by $h^0(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma)$ and the multiplets (Q, λ_α) whose number is determined by $h^1(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma)$ (or $h^0(\Sigma, K_\Sigma^{1/2} \otimes V^\vee|_\Sigma)$). Here $V|_\Sigma$ is V restricted to Σ. The fields (Q, λ_α) transform in the fundamental representation of $\mathrm{SU}(N_c)$ and the fields $(\tilde{Q}, \tilde{\lambda_\alpha})$ transform in the antifundamental representation of $\mathrm{SU}(N_c)$. To generate SQCD, we need the number of fundamental and antifundamental multiplets to be the same. This means that the Euler characteristic

$$\chi(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) = h^0(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) - h^1(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) \tag{3.18}$$

has to vanish. From the Riemann-Roch theorem (see, for example, [47]) it follows that

$$\begin{aligned} \chi(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) &= (1-g)c_0(K_\Sigma^{1/2} \otimes V|_\Sigma) + c_1(K_\Sigma^{1/2} \otimes V|_\Sigma) \\ &= (1-g)c_0(V|_\Sigma) + c_1(V|_\Sigma) + c_0(V|_\Sigma)c_1(K_\Sigma^{1/2}), \end{aligned} \tag{3.19}$$

where g is the genus of Σ. In this paper, we will consider the case

$$\Sigma \simeq \mathbb{P}^1. \tag{3.20}$$

Then, taking into account that

$$K_\Sigma^{1/2} \simeq \mathcal{O}(-1) \tag{3.21}$$

it follows from (3.19) that $\chi(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) = 0$ if

$$c_1(V|_\Sigma) = 0. \tag{3.22}$$

This condition is trivially satisfied if V has structure group $\mathrm{SU}(n)$.

Thus, the question of analyzing how to obtain SQCD with appropriate number of light fields is reduced to analyzing under what conditions the vector bundle

$$\mathcal{O}(-1) \otimes V|_\Sigma \tag{3.23}$$

has global holomorphic sections. This problem is known to arise in a different context, namely, under what conditions the non-perturbative superpotential due to a string (open membrane) instanton in heterotic M-theory does or does not vanish [29 − 32]. In that context, Σ is an isolated sphere inside the Calabi-Yau threefold on which the $E_8 \times E_8$ heterotic string theory is compactified, V is a vector bundle on the threefold and $h^0(\Sigma, \mathcal{O}(-1) \otimes V|_\Sigma)$ counts the number of the zero modes of the Dirac operator coupled to the world-sheet fermions on Σ. The existence or non-existence of the global sections of $\mathcal{O}(-1) \otimes V|_\Sigma$ depends on the moduli of V and on the complex structure of the Calabi-Yau threefold. This problem was analyzed in detail for some geometries in [31, 32] where the dependence of the non-perturbative superpotential on the vector bundle moduli was explicitly calculated. In the next section we will specify the details of our examples so that the set-up is reduced to the one studied in [31, 32]. This will allow us to calculate the locus in the moduli space near which the right number of the fundamental fields becomes light, realizing this way SQCD in the free magnetic range.

To finish this section, let us discuss the quadratic superpotential for the (anti)-fundamental fields Q and $\tilde{Q}$. As was shown in [22], the action I_Σ contains terms which give rise to the superpotential. This superpotential can be written as follows. Let ω_Q be the element of $H^0(\Sigma, K_\Sigma^{1/2} \otimes V^\vee|_\Sigma)$ corresponding to Q. That is, the world-volume field σ is written as

$$\sigma = \sum Q \cdot \omega_Q, \tag{3.24}$$

where the sum is over all zero modes of σ. Similarly, let $\omega_{\tilde{Q}}$ be the element in $H^0(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma)$ corresponding to $\tilde{Q}$. Note that

$$\omega_Q \cdot \omega_{\tilde{Q}} \in H^0(\Sigma, K_\Sigma \otimes (V \otimes V^\vee)|_\Sigma) \tag{3.25}$$

At last, let ω_ϕ be the differential form corresponding to the vector bundle modulus ϕ. It is a standard result that[3]

$$\omega_\phi \in H^1(\mathcal{S}, EndV) = H^1(\mathcal{S}, V \otimes V^\vee) \tag{3.26}$$

If we restrict eq. (3.26) to Σ and use the Serre dulity

$$H^1(\Sigma, (V \otimes V^\vee)|_\Sigma) \simeq H^0(\Sigma, K_\Sigma^{-1/2} \otimes (V \otimes V^\vee)|_\Sigma)^\vee \tag{3.27}$$

we that one can pair up elements in (3.25) and (3.27) to obtain a complex number since they parametrize the spaces dual to each other. That is, we have the following natural map

$$H^0(\Sigma, K_\Sigma^{1/2} \otimes V|_\Sigma) \otimes H^1(\Sigma, (V \otimes V^\vee)|_\Sigma) \otimes H^0(\Sigma, K_\Sigma^{1/2} \otimes V^\vee|_\Sigma) \to \mathbb{C}. \tag{3.28}$$

Explicitly, it can be done as follows. We have

$$\omega_Q \cdot \omega_{\tilde{Q}} \in H^0(\Sigma, K_\Sigma \otimes (V \otimes V^\vee)|_\Sigma) \simeq H^{(1,0)}_{\bar{\partial}}(\Sigma, (V \otimes V^\vee)|_\Sigma) \tag{3.29}$$

[3]See, for example, section 15.7.3 of [45].

Hence, we can view $\omega_Q \cdot \omega_{\tilde{Q}}$ as a $(1,0)$ differential form on Σ. On the other hand,

$$\omega_\phi \in H^1(\Sigma, (V \otimes V^\vee)|_\Sigma) \simeq H^{(0,1)}_{\bar{\partial}}(\Sigma, (V \otimes V^\vee)|_\Sigma). \tag{3.30}$$

Hence, we can view ω_ϕ as a $(0,1)$-differential form on Σ. Thus, the superpotential can be written as

$$W = \lambda \phi \mathrm{Tr}(Q \cdot \tilde{Q}), \tag{3.31}$$

where

$$\lambda = \int_\Sigma \omega_Q \cdot \omega_{\tilde{Q}} \wedge \omega_\phi, \tag{3.32}$$

where we suppressed the flavor indices.

3.5 The summary of the model

In this small subsection, we will summarize the ingredients necessary to build SQCD found above. The various pieces of the spectrum come from the three different sources, the surface $\mathcal{S}'$, the surface $\mathcal{S}$ and the intersection curve Σ. More precisely the role of each of them is as follows.

- The surface $\mathcal{S}'$ contributes $\mathcal{N} = 1$, $\mathrm{SU}(N_c)$ gauge theory.
- The surface $\mathcal{S}$ contributes vector bundle moduli.
- The intersection curve Σ contributes matter fields Q and $\tilde{Q}$ in the fundamental and antifundamental representations of the gauge group $\mathrm{SU}(N_c)$. The mass of these fields is controlled by the vector bundle moduli through the superpotential (3.31).

4. An F-theory realization of SQCD in the free magnetic range

4.1 The geometric data

In this section, we will give a realization of the ideas developed in the previous section. From the above consideration it follows that the details of the geometry of the surface $\mathcal{S}'$ are irrelevant. The role of it is to produce the $\mathrm{SU}(N_c)$ vector multiplet. Therefore, we only need to specify the surface $\mathcal{S}$ and the curve $\Sigma \subset \mathcal{S}$. Motivated by the heterotic-F-theory duality it is reasonable to choose $\mathcal{S}$ to be the base of an elliptically fibered Calabi-Yau threefold. We will choose $\mathcal{S}$ to be rational elliptic surface dP_9 which is known to a be a possible base for a Calabi-Yau threefold. Various properties of dP_9 can be found, for example, in [46]. It is worth pointing out that elliptically fibered over dP_9 Calabi-Yau threefolds as well as their quotient over a discrete group are often used in GUT and Standard Model heterotic compactifications. In particular, such manifolds were used in constructing a heterotic standard model in [11 − 16].

Let us present here some facts about dP_9. The surface dP_9 is obtained from $\mathbb{P}^2$ by blowing up nine distinct points. Thus, the basis of effective curves in dP_9 can be chosen to be

$$\{\ell, e_1, \ldots e_9\}, \tag{4.1}$$

where ℓ is the hyperplane divisor inherited from $\mathbb{P}^2$ and $e_1, \ldots e_9$ are the nine exceptional divisors each isomorphic to $\mathbb{P}^1$. However, it is more convenient to work with a different basis. The surface dP_9 admits an elliptic fibration over $\mathbb{P}^1$. We identify the base of this fibration, σ, with one of the exceptional curves, say e_1. Let π be the projection map

$$\pi : dP_9 \to \sigma = e_1. \tag{4.2}$$

A more convenient basis is

$$\{F, e_1, \ldots e_9\}, \tag{4.3}$$

where F is the class of the elliptic fiber. In terms of the curves in the basis (4.1) it is given by

$$F = 3\ell - \sum_{i=1}^{9} e_i. \tag{4.4}$$

The intersection numbers of the curves in (4.3) are given by

$$e_i \cdot e_j = -\delta_{ij}, \quad e_i \cdot F = 1. \tag{4.5}$$

The Chern classes of dP_9 are given by

$$c_1(dP_9) = F, \quad c_2(dP_9) = 12. \tag{4.6}$$

Being an elliptic fibration, dP_9 can be described by the Weierstrass equation

$$y^2 z = x^3 + f x z^2 + g z^3, \tag{4.7}$$

where f and g are polynomials on the base $\sigma \simeq \mathbb{P}^1$. More precisely, f is a polynomial of degree four and g is a polynomial of degree six. Furthermore, z, x and y are sections of the following line bundles [5]

$$z \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma)), \quad x \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma + 2F)), \quad y \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma + 3F)). \tag{4.8}$$

One can see that each term in eq (4.7) is a section of the same line bundle.

After having specified the surface $\mathcal{S}$, we need to specify a genus zero curve $\Sigma \in \mathcal{S}$. We will choose it to be the base of dP_9, σ. That is,

$$\Sigma = \sigma \simeq \mathbb{P}^1. \tag{4.9}$$

The next ingredient which needs to be specified is an SU(n) instanton V on dP_9. Since dP_9 is elliptically fibered we can use the spectral cover construction [5, 33]. According to this construction, an SU(n) vector bundle on elliptic dP_9 (or any elliptically fibered manifold) can be obtained from the spectral data

$$(\mathcal{C}, \mathcal{N}), \tag{4.10}$$

where the spectral cover $\mathcal{C}$ is an n-fold cover of the base σ (in our case $\mathcal{C}$ is a Riemann surface) and $\mathcal{N}$ is a line bundle on $\mathcal{C}$. The corresponding SU(n) vector bundle V can be

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obtained from the spectral data (4.10) by a Fourier-Mukai transformation [5, 33]. We will choose the homology class of the spectral cover to be of the form

$$\mathcal{C} = n\sigma + kF. \tag{4.11}$$

The coefficient n determines the rank of the vector bundle V and the coefficient k can be shown to be the second Chern class (the instanton number) of V. In order to make sure that the bundle V is stable (that is, admits a connection solving the BPS equations (2.12)) the homology class of the spectral cover has to contain irreducible curves. One can show that this is the case if the following condition is satisfied

$$k \geq n. \tag{4.12}$$

Let us work out some properties of $\mathcal{C}$. As we mentioned before, $\mathcal{C}$ is simply a Riemann surface. For future reference, let us calculate its genus $g_{\mathcal{C}}$. It can be obtained using the adjunction and Riemann-Hurwitz formulas (see for example [47]). From the adjunction formula it follows that the canonical bundle of $\mathcal{C}$ is

$$K_{\mathcal{C}} = \big(K_{dP_9} \otimes \mathcal{O}_{dP_9}(\mathcal{C})\big)|_{\mathcal{C}}. \tag{4.13}$$

Therefore the degree of the canonical bundle of $\mathcal{C}$ is given by

$$\mathrm{deg} K_{\mathcal{C}} = K_{dP_9} \cdot \mathcal{C} + \mathcal{C} \cdot \mathcal{C}. \tag{4.14}$$

Knowing the degree of the canonical bundle we can obtain the genus by the Riemann-Hurwitz formula

$$2g_{\mathcal{C}} - 2 = \mathrm{deg} K_{\mathcal{C}}. \tag{4.15}$$

Then from eqs. (4.5), (4.6), (4.11), (4.14) and (4.15) it follows that

$$g_{\mathcal{C}} = nk - \frac{(n-1)(n+2)}{2}. \tag{4.16}$$

Now we will calculate how many parameters the linear system of $\mathcal{C}$ has. The number of *projective* parameters of the spectral cover is given by

$$h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{C})). \tag{4.17}$$

This number can be calculated using a simple Leray spectral sequence according to which

$$h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{C})) = h^0(\sigma, \pi_* \mathcal{O}_{dP_9}(\mathcal{C})) = h^0(\sigma, \pi_* \mathcal{O}_{dP_9}(n\sigma + kF)). \tag{4.18}$$

The direct image $\pi_* \mathcal{O}_{dP_9}(n\sigma + kF)$ was computed in appendix C of [32] by induction in n. Here we just quote the result

$$\pi_* \mathcal{O}_{dP_9}(n\sigma + kF) = \mathcal{O}(k) \oplus \bigoplus_{i=2}^{n} \mathcal{O}(k-i). \tag{4.19}$$

Note that for $k \geq n$ all entries in the right hand side of (4.19) are non-negative. Since $h^0(\mathbb{P}^1, \mathcal{O}(i)) = i+1$ for $i \geq 0$, it follows that

$$h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{C})) = (k+1) + (k-1) + \cdots + (k-n+1) = nk - \frac{(n+1)(n-2)}{2}. \quad (4.20)$$

The parameters of the spectral cover form a projective space $\mathbb{P}^{h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{C}))-1}$ [48, 49]. Later, we will introduce an explicit coordinate parametrization of this space.

Now we move on to discussing the line bundle $\mathcal{N}$. An arbitrary choice of $\mathcal{N}$ on the spectral cover $\mathcal{C}$ will lead to a vector bundle V on dP_9 with structure group $\mathrm{U}(n)$. The condition under which $\mathcal{N}$ produces an $\mathrm{SU}(n)$ rather than $\mathrm{U}(n)$ vector bundle was derived in [5]. It can be formulated as follows: the degree of $\mathcal{N}$ has to be related to the genus of the spectral cover as follows

$$\deg \mathcal{N} = g_{\mathcal{C}} - 1 + n. \quad (4.21)$$

The moduli space of line bundles $\mathcal{N}$ is the Jacobian $\mathcal{J}_{g_{\mathcal{C}}} \simeq (\mathbb{T}^2)^{g_{\mathcal{C}}}$. Thus, the moduli space of the vector bundle V is a Jacobian bundle over the projective space $\mathbb{P}^{h^0(dP_9, \mathcal{O}_{dP_9}(\mathcal{C}))-1}$. Unfortunately, it is very difficult to introduce an explicit parametrization of the Jacobian and have an analytic control over it. Therefore, at this step, we will simplify our analysis. We will fix the moduli of $\mathcal{N}$ at some particular values and study how $h^0(\sigma, K_\sigma^{1/2} \otimes V|_\sigma)$ behaves only as we move in the projective space of the parameters of the spectral cover. We will fix the line bundles $\mathcal{N}$ on $\mathcal{C}$ as follows. We will choose $\mathcal{N}$ to be the restriction of the following discrete line bundles on dP_9.

$$\mathcal{N} = \mathcal{O}_{dP_9}\left(n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)kF + \left(\frac{1}{2} + n\lambda\right)F\right), \quad (4.22)$$

where the discrete parameter λ has to be chosen in such a way that the right hand side in (4.22) is an integral class on dP_9. For example, if n is odd one always gets an integral class if λ is half-integer. Starting this point, we will always mean by $\mathcal{N}$ a line bundle on dP_9 of the form (4.22) and the corresponding spectral line bundle on $\mathcal{C}$ we will denote as $\mathcal{N}|_{\mathcal{C}}$. It is straightforward to check using eqs. (4.11), (4.5) and (4.13) that the degree of $\mathcal{N}|_{\mathcal{C}}$ is indeed given by eq. (4.21) independent of λ.

To summarize, we will consider vector bundles V on dP_9 constructed using the spectral data ($\mathcal{C}$, $\mathcal{N}|_{\mathcal{C}}$), where $\mathcal{C}$ is given by eq. (4.11) and $\mathcal{N}|_{\mathcal{C}}$ is obtained by restriction of (4.22) to $\mathcal{C}$.

4.2 The matter localized on the curve

In this subsection, we will consider the matter localized on the curve σ. As was discussed before, it is determined by the cohomology groups

$$H^0(\sigma, \mathcal{O}(-1) \otimes V|_\sigma), \quad H^1(\sigma, \mathcal{O}(-1) \otimes V|_\sigma), \quad (4.23)$$

where we have used the fact that $K_\sigma^{1/2} = \mathcal{O}(-1)$. The analysis in this subsection will be similar to the one in [31, 32] though the context is different. Our goal is to derive the equation in the moduli space of $\mathcal{C}$ along the zero locus of which one gets massless fundamental fields whereas away from this locus all the fundamental fields are massive.

The bundle $V|_\sigma$ can be obtained from the spectral data $(\mathcal{C}, \mathcal{N}|_\mathcal{C})$ as follows [31, 32]

$$V|_\sigma = \pi_\mathcal{C} \mathcal{N}|_\mathcal{C}, \tag{4.24}$$

where $\pi_\mathcal{C} : \mathcal{C} \to \sigma$ is the n-fold cover map. Then from a Leray spectral sequence it follows that

$$h^0(\sigma, \mathcal{O}(-1) \otimes V|_\sigma) = h^0(\mathcal{C}, (\mathcal{N} \otimes \mathcal{O}_{dP_9}(-F))|_\mathcal{C}). \tag{4.25}$$

Denote

$$\mathcal{N}(-F) = \mathcal{N} \otimes \mathcal{O}_{dP_9}(-F). \tag{4.26}$$

Thus, we have to study under what conditions $h^0(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C})$ vanishes. Note that the Euler characteristic of $\mathcal{N}(-F)|_\mathcal{C}$ vanishes. Indeed, from the Riemann-Roch formula

$$\chi(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C}) = d - g_\mathcal{C} + 1, \tag{4.27}$$

where by d we denoted the degree of the line bundle $\mathcal{N}(-F)|_\mathcal{C}$. Since the degree of $\mathcal{N}|_\mathcal{C}$ is $g_\mathcal{C} - 1 + n$ it follows that

$$d = g_\mathcal{C} - 1 \tag{4.28}$$

and, hence, the Euler characteristic in (4.26) vanishes.

The dimension $h^0(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C})$ depends on the parameters of $\mathcal{C}$. As we move in the projective space of these parameters, $h^0(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C})$ might jump. We are interested in examples where $h^0((\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C})$ is zero at a generic point in the moduli space and jumps along some subvariety. Let us now show how to derive the equation of this subvariety. First, we will give some general discussion and then give a specific example.

Consider the following short exact sequence on dP_9

$$0 \to E \otimes \mathcal{O}_{dP_9}(-D) \stackrel{f_D}{\to} E \stackrel{r}{\to} E|_D \to 0. \tag{4.29}$$

Here E is an arbitrary holomorphic vector bundle on dP_9 and D is a divisor in it. The map r is just the restriction map. The map f_D is a multiplication by a section of $\mathcal{O}_{dP_9}(D)$ which vanishes precisely on D. This sequence can be understood as follows. Let e be any section of E. Let us restrict e to D and find the kernel of the restriction map. The kernel consists of such sections e which vanish on D. Such sections can be written as $e = f_D e'$ for some e'. It is clear that e' transforms with transition functions of $E \otimes \mathcal{O}_{dP_9}(-D)$. This means that the kernel of r is $E \otimes \mathcal{O}_{dP_9}(-D)$. For our purposes, we choose

$$E = \mathcal{N}(-F), \quad D = \mathcal{C}. \tag{4.30}$$

The sequence (4.29) becomes

$$0 \to \mathcal{N}(-F - \mathcal{C}) \stackrel{f_\mathcal{C}}{\to} \mathcal{N}(-F) \stackrel{r}{\to} \mathcal{N}(-F)|_\mathcal{C} \to 0, \tag{4.31}$$

where by $\mathcal{N}(-F - \mathcal{C})$ we simply denoted $\mathcal{N}(-F) \otimes \mathcal{O}_{dP_9}(-\mathcal{C})$. From here we obtain the corresponding long exact sequence of the cohomology groups

$$\begin{aligned} 0 \to H^0(dP_9, \mathcal{N}(-F - \mathcal{C})) &\to H^0(dP_9, \mathcal{N}(-F)) \to H^0(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C}) \\ &\to H^1(dP_9, \mathcal{N}(-F - \mathcal{C})) \to H^1(dP_9, \mathcal{N}(-F)) \to H^1(\mathcal{C}, \mathcal{N}(-F)|_\mathcal{C}) \to \dots . \end{aligned} \tag{4.32}$$

Note that the cohomology group $H^0(\mathcal{C}, \mathcal{N}(-F)|_{\mathcal{C}})$ is exactly the object we are interested in. Also note that if $h^0(dP_9, \mathcal{N}(-F))$ is non-zero, $h^0(\mathcal{C}, \mathcal{N}(-F)|_{\mathcal{C}})$ cannot vanish. Hence, in this case we always have massless fundamental matter. Therefore, we will study the case when $h^0(dP_9, \mathcal{N}(-F)) = 0$. Then the sequence (4.32) simplifies and becomes

$$0 \to H^0(\mathcal{C}, \mathcal{N}(-F)|_{\mathcal{C}}) \to W_1 \xrightarrow{f_{\mathcal{C}}} W_2 \to \ldots, \tag{4.33}$$

where W_1 and W_2 are the following vector spaces

$$W_1 = H^1(dP_9, \mathcal{N}(-F-\mathcal{C})) \tag{4.34}$$

and

$$W_2 = H^1(dP_9, \mathcal{N}(-F)). \tag{4.35}$$

Both W_1 and W_2 are finite-dimensional vector spaces. The map $f_{\mathcal{C}}$ in (4.33) between them is a multiplication by a section of $\mathcal{O}_{dP_9}(\mathcal{C})$. It depends on the parameters of $\mathcal{C}$. This map can be organized as a finite-dimensional matrix. Thus, $h^0(\mathcal{C}, \mathcal{N}(-F)|_{\mathcal{C}})$ is non-zero if the matrix $f_{\mathcal{C}}$ has a non-trivial kernel. We will consider the case when $\mathrm{dim} W_1 = \mathrm{dim} W_2$. Then $f_{\mathcal{C}}$ is a square matrix. Therefore, $h^0(\mathcal{C}, \mathcal{N}(-F)|_{\mathcal{C}})$ is non-zero if and only if

$$\det f_{\mathcal{C}} = 0. \tag{4.36}$$

Away from the locus given by eq. (4.36) all fundamental fields Q and $\tilde{Q}$ are very massive and the theory on the intersecting seven-branes is just pure $\mathrm{SU}(N_c)$ supersymmetric Yang-Mills theory. Near the locus (4.36) some number of the fundamental matter fields becomes light and the theory is SQCD with massive matter. This equation alone does not tell us exactly how many fundamental fields we obtain. We will discuss it later in this section. Now we will present an example of computation of $\det f_{\mathcal{C}}$ [31, 32].

Example. In this example, we will choose a vector bundle V to have the structure group SU(3). We will specify the second Chern class of V to be $k = 5$. In addition, we choose the discrete parameter λ in (4.22) to be $\lambda = \frac{3}{2}$. Then we obtain

$$\begin{aligned} \mathcal{C} &= 3\sigma + 5F, \\ \mathcal{N}(-F) &= \mathcal{O}_{dP_9}(6\sigma - F), \\ \mathcal{N}(-F-\mathcal{C}) &= \mathcal{O}_{dP_9}(3\sigma - 6F). \end{aligned} \tag{4.37}$$

Let us start with the explicit parametrization of the spectral cover. The number of the *projective* parameters is given by eq. (4.20). In our case it is 13. Since from eq. (4.19) we have

$$\pi_* \mathcal{O}_{dP_9}(3\sigma + 5F) = \mathcal{O}(5) \oplus \mathcal{O}(3) \oplus \mathcal{O}(2), \tag{4.38}$$

we can write the equation for $\mathcal{C}$ as follows

$$\mathcal{C} = a_5 z + a_3 x + a_2 y, \tag{4.39}$$

where z, x, y are the variables in the Weierstrass equation (4.7), (4.8). The coefficients a_k are $a_k = \pi^* A_k$, where A_k is a section of $H^0(\sigma, \mathcal{O}(k))$, that is a polynomial of degree k on

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$\sigma \simeq \mathbb{P}^1$. Thus, if (u, v) are projective coordinates on σ then we have the following explicit parametrization of A_k

$$\begin{aligned} A_5 &= \psi_1 u^5 + \psi_2 u^4 v + \psi_3 u^3 v^2 + \psi_4 u^2 v^3 + \psi_5 u v^4 + \psi_6 v^5, \\ A_3 &= \phi_1 u^3 + \phi_2 u^2 v + \phi_3 u v^2 + \phi_4 v^3, \\ A_2 &= \chi_1 u^2 + \chi_2 u v + \chi_3 v^3, \end{aligned} \tag{4.40}$$

where $\{\psi_a, \phi_b, \chi_c\}$ are the 13 *projective* parameters of the spectral cover. The actual equation of $\mathcal{C}$ in dP_9 is obtained by setting (4.39) to zero. This equation is invariant under rescaling of all $\{\psi_a, \phi_b, \chi_c\}$ by a non-zero complex number. Therefore, only 12 parameters are independent. They parametrize the projective space $\mathbb{P}^{12}$.

In the next step, we need to parametrize the vector spaces W_1 and W_2. The idea is to push W_1 and W_2 down to the base $\sigma \simeq \mathbb{P}^1$ where one can use a paramerization in terms of polynomials. From a Leray spectral sequence it follows that

$$W_1 = H^1(dP_9, N(-F - \mathcal{C})) \simeq H^1(\sigma, \pi_* \mathcal{N}(-F - \mathcal{C})). \tag{4.41}$$

To obtain this result we used the fact that $R^1 \pi_* \mathcal{N}(-F - \mathcal{C}) = 0$ which follows from the explicit form of $\mathcal{N}(-F - \mathcal{C})$ in eq. (4.37). Indeed, by definition, the sheaf $R^1 \pi_* \mathcal{N}(-F - \mathcal{C})$ is generated at each point p on σ by the cohomology of the fiber $H^1(F_p, \mathcal{N}(-F - \mathcal{C})|_{F_p})$, where F_p is the elliptic fiber over p. From eq. (4.37) and intersection numbers (4.5) it follows that the degree of $\mathcal{N}(-F - \mathcal{C})|_{F_p}$ is 3 which is positive. Then it follows from the Kodaira vanishing theorem [47] that $H^1(F_p, \mathcal{N}(-F - \mathcal{C})|_{F_p}) = 0$. Thus, $R^1 \pi_* \mathcal{N}(-F - \mathcal{C})$ is the zero sheaf. To continue, from eqs. (4.37) and (4.19) we find that

$$\pi_* \mathcal{N}(-F - \mathcal{C}) = \mathcal{O}(-6) \oplus \mathcal{O}(-8) \oplus \mathcal{O}(-9). \tag{4.42}$$

Since $h^1(\mathbb{P}^1, \mathcal{O}(-i)) = i - 1$ for positive i, we find that the dimension of W_1 is

$$\dim W_1 = 5 + 7 + 8 = 20. \tag{4.43}$$

Moreover, the decomposition (4.42) allows us to parametrize the elements of W_1 in terms of the differentials on σ. Let $B_{-i} \in H^1(\sigma, \mathcal{O}(-i))$, $i = 6,\ 8,\ 9$ be the differentials on σ. Let $b_{-i} = \pi^* B_{-i}$ be their pullback to dP_9. B_{-i} are elements of $H^1(dP_9, \mathcal{O}_{dP_9}(-iF))$. To construct an element $w_1 \in W_1$ we need to multiply $\pi^* B_{-6}$ by a section of $\mathcal{O}_{dP_9}(3\sigma)$, $\pi^* B_{-8}$ by a section of $\mathcal{O}_{dP_9}(3\sigma + 2F)$ and $\pi^* B_{-9}$ by a section of $\mathcal{O}_{dP_9}(3\sigma + 3F)$. We can choose these sections to be z, x and y. Thus, $w_1 \in W_1$ can be parametrized as

$$w_1 = b_{-6} z + b_{-8} x + b_{-9} y. \tag{4.44}$$

Similarly, one can parametrize $w_2 \in W_2$. First, we note that

$$\begin{aligned} W_2 &= H^1(dP_9, \mathcal{N}(-F)) \simeq H^1(\sigma, \pi_* \mathcal{N}(-F)) \\ &= \mathcal{O}(-1) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6) \oplus \mathcal{O}(-7). \end{aligned} \tag{4.45}$$

The dimension of W_2 is then given by

$$\dim W_2 = 0 + 2 + 3 + 4 + 5 + 6 = 20. \tag{4.46}$$

Now an element $w_2 \in W_2$ can be written as follows

$$w_2 = c_3 zx + c_4 zy + c_5 x^2 + c_6 xy + c_7 y^2, \tag{4.47}$$

where $c_{-j} = \pi^* C_{-j}$, $j = 3, 4, 5, 6, 7$, where C_{-j} are elements of $H^1(\sigma, \mathcal{O}(-j))$, that is differentials on σ. The map $f_{\mathcal{C}}$ is a multiplication of w_1 in eq. (4.44) by $\mathcal{C}$ in eq. (4.39). The result of it must be an element w_2 in eq. (4.47). This multiplication can be organized in a 20×20 matrix depending on $\{\psi_a, \phi_b, \chi_c\}$ in (4.40). We present some details of construction of this matrix in appendix C. The determinant of this matrix is

$$\det f_{\mathcal{C}} = \mathcal{P}^4, \tag{4.48}$$

where $\mathcal{P}$ is a homogeneous polynomial of degree 5

$$\begin{aligned} \mathcal{P} = \; & \chi_1^2\chi_3\phi_3^2 - \chi_1^2\chi_2\phi_3\phi_4 - 2\chi_1\chi_3^2\phi_3\phi_1 \\ & -\chi_1\chi_2\chi_3\phi_3\phi_2 + \chi_2^2\chi_3\phi_1\phi_3 + \phi_4^2\chi_1^3 \\ & -2\phi_2\phi_4\chi_3\chi_1^2 + \chi_1\chi_3^2\phi_2^2 + 3\phi_1\phi_4\chi_1\chi_2\chi_3 \\ & +\phi_2\chi_1\phi_4\chi_2^2 + \phi_1^2\chi_3^3 - \phi_2\chi_2\phi_1\chi_3^2 - \phi_4\phi_1\chi_2^3. \end{aligned} \tag{4.49}$$

Note that $\mathcal{P}$ does not depend on ψ_a. Eqs. (4.48), (4.49) represent an explicit equation in the moduli space of the vector bundle near which some number of (anti)-fundamental multiplets becomes light. Note that, this is not enough to generate massive SQCD in the free magnetic range since we need to know how many multiplets become light. We will analyze it later in this section. Before that, we will show that the reason why the (anti)-fundamental multiplets are massive away from the zero locus of $\det f_{\mathcal{C}}$ is precisely the Yukawa-type superpotential (3.31), (3.32).

4.3 The superpotential

In this subsection, we will show that the exact sequence (4.32) which we can write as

$$0 \to H^0(\sigma, K_\sigma^{1/2} \otimes V|_\sigma) \to W_1 \xrightarrow{f_{\mathcal{C}}} W_2 \to H^1(\sigma, K_\sigma^{1/2} \otimes V|_\sigma) \to \dots \tag{4.50}$$

can be interpreted as an algebraic geometry version of the superpotential (3.31), (3.32). First, we will use the Serre duality to write

$$H^1(\sigma, K_\sigma^{1/2} \otimes V|_\sigma) \simeq H^0(\sigma, K_\sigma^{1/2} \otimes V^\vee|_\sigma)^\vee. \tag{4.51}$$

Second, from (4.50) it follows that $H^0(\sigma, K_\sigma^{1/2} \otimes V|_\sigma)$ is a subgroup of W_1. Similarly, $H^0(\sigma, K_\sigma^{1/2} \otimes V^\vee|_\sigma)$ is a subgroup of $W_2^\vee$. When we multiply an element $w_1 \in W_1$ by $f_{\mathcal{C}}$ we obtain an element $w_2 \in W_2$. This element can be paired up with an element of $W_2^\vee$ to produce a complex number. The map $f_{\mathcal{C}}$ depends on the vector bundle moduli and, hence, can be viewed as an element in $H^1(\sigma, (V \otimes V^\vee)|_\sigma)$. Thus, the sequence (4.50) gives a natural map

$$H^0(\sigma, K_\sigma^{1/2} \otimes V|_\sigma) \otimes H^1(\sigma, (V \otimes V^\vee)|_\sigma) \otimes H^0(\sigma, K_\sigma^{1/2} \otimes V^\vee|_\sigma) \to \mathbb{C}. \tag{4.52}$$

This map is exactly the superpotential as explained at the end of subsection 3.4.

4.4 Examples of SQCD

In this final subsection, we will give examples of SQCD in the free magnetic range within the framework of the **Example.** given in subsection 4.2. For this we need to understand how many (anti)-fundamental flavors become light near the locus $\det f_{\mathcal{C}} = 0$ in eqs.(4.48), (4.49). This number is the dimension of the kernel of the matrix $f_{\mathcal{C}}$. At any point in the moduli space where $\det f_{\mathcal{C}} = 0$ the rank of the matrix $f_{\mathcal{C}}$ is less than 20. Note that the rank changes as we move in the zero locus of $\det f_{\mathcal{C}}$. Let r be the rank of $f_{\mathcal{C}}$ at some point in the moduli space. Then the dimension of the kernel of $f_{\mathcal{C}}$ is simply $20 - r$. Unfortunately, a detailed study of the rank of $f_{\mathcal{C}}$ in different regimes in the moduli space requires a hard numeric work. However, for some values of the moduli ψ_a, ϕ_b, χ_c the matrix $f_{\mathcal{C}}$ simplifies and one can prove the existence of subspaces where a certain specific number of flavors becomes light. It will be enough to present examples of SQCD in the free magnetic range.

Let us study the subspace of $\mathbb{P}^{12}$ where $\psi_a = 0$, $a = 1, \dots, 6$. Then one can show that it is possible to arrange the rows and columns in such a way that the matrix $f_{\mathcal{C}}$ becomes block-diagonal with four identical 5×5 blocks of the form

$$M = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 \\ 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \chi_1 & \chi_2 & \chi_3 & 0 & 0 \\ 0 & \chi_1 & \chi_2 & \chi_3 & 0 \\ 0 & 0 & \chi_1 & \chi_2 & \chi_3 \end{pmatrix}. \tag{4.53}$$

Note that the determinant of M is precisely the polynomial $\mathcal{P}$ in eq. (4.49). In other words, the determinant of the whole matrix $f_{\mathcal{C}}$ is the determinant of M raised to the power four. It is easy to see that setting, for example, $\phi_1 = \chi_1 = \chi_2 = 0$ reduces the rank of the matrix M by one. Since $f_{\mathcal{C}}$ consists of four blocks of M the rank of $f_{\mathcal{C}}$ at this locus drops by four. This proves that there exist a subvariety $\mathbb{L}_1 \subset \mathbb{P}^{12}$ containing the subspace

$$\psi_a = \phi_1 = \chi_1 = \chi_2 = 0, \quad a = 1, \dots, 6, \tag{4.54}$$

where the rank of the matrix $f_{\mathcal{C}}$ drops by four. Similarly, it is not difficult to prove that there exists a subvarity $\mathbb{L}_2 \in \mathbb{P}^{12}$ where the rank of M drops by two and the rank of $f_{\mathcal{C}}$ drops by eight. For example, the following subspace is contained in $\mathbb{L}_2$

$$\psi_a = \phi_1 = \phi_2 = \chi_1 = \chi_2 = 0, \quad a = 1, \dots, 6. \tag{4.55}$$

Of course, the subvarieties $\mathbb{L}_1$ and $\mathbb{L}_2$ are much wider than the their subspaces specified in eqs. (4.54) and (4.55). However, for our purposes it is enough to establish that $\mathbb{L}_1$ and $\mathbb{L}_2$ are non-empty. It is very likely that by turning on the moduli ψ_a one can achieve that the rank of $f_{\mathcal{C}}$ drops by any number between 4 and 8. Now we give some examples of SQCD in the free magnetic range.

- Let us choose the low-energy gauge group to be

$$\Gamma_{\mathcal{S}'} = \mathrm{SU}(3). \tag{4.56}$$

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The free magnetic range for the gauge group SU(3) is given by

$$4 \leq N_f < \frac{9}{2}. \tag{4.57}$$

We see that $N_f = 4$ is a solution to (4.57). We showed above that there exists a subvariety $\mathbb{L}_1$ in the moduli space where the rank of $f_{\mathcal{C}}$ drops by four. This means that dimension of the kernel of $f_{\mathcal{C}}$ is four. This, in turn, means that near $\mathbb{L}_1$ we have exactly four fundamental flavors. Thus, near a generic point of the subvariety $\mathbb{L}_1$ we generate SQCD in the free magnetic range with

$$N_c = 3, \quad N_f = 4. \tag{4.58}$$

- Let us now choose the low-energy gauge group to be

$$\Gamma_{\mathcal{S}'} = \mathrm{SU}(6). \tag{4.59}$$

The free magnetic range for the gauge group SU(6) is given by

$$7 \leq N_f < 9. \tag{4.60}$$

From our discussion earlier in this subsection we know that there exists a subvariety $\mathbb{L}_2$ in the moduli space where the rank of the matrix $f_{\mathcal{C}}$ drops by eight. Hence, the dimension of the kernel of $f_{\mathcal{C}}$ becomes eight. Thus, near $\mathbb{L}_2$ we have exactly eight fundamental flavors. This ways we generate SQCD in the free magnetic range with

$$N_c = 6, \quad N_f = 8. \tag{4.61}$$

Note that the fact that $\det f_{\mathcal{C}}$ is given by a polynomial of high degree is rather helpful in generating a suitable number of flavors.

Clearly, using the technics presented in this paper, one can construct many other examples of SQCD on F-theory seven-branes and find the regimes in the moduli space where the number of flavors is in the free magnetic range.

5. Conclusion

In this paper, we address the question of realizing dynamically SUSY breaking SQCD [25] in F-theory. Our starting point is the field theory on the intersecting seven-branes obtained by Beasley, Heckman and Vafa in [22]. In our model, one of the seven-branes realizes $\mathcal{N} = 1$, SU(N_c) supersymmetric Yang-Mills theory. The other one contributes vector bundle moduli. Finally, the matter fields in the (anti)-fundamental representation of SU(N_c) comes the intersection. These matter fields have a quadratic superpotential with the mass matrix depending on the vector bundle moduli. In order to obtain SUSY breaking SQCD in the free magnetic range one has to move to a certain regime in the moduli space where an appropriate number of the matter fields becomes light. Conceptually, this is similar to analyzing how many Higgs multiplets one has in heterotic standard models of [11 – 16]. For example, in the model of [15] one can have zero, one or two Higgs multiplets depending on the location in the moduli space. Though in this paper, for concreteness, we work in the context of some specific choices of the type of the ADE-singularity and of the geometric data, our method has, of course, a wider applicability.

A natural question which arises is whether it is possible to generate the mass term for the (anti)-fundamental multiplets not by vector bundle moduli but by $D1$- or $D3$-brane instantons. The mass obtained this way will be exponentially suppressed by the volume of the Euclidean D-brane. This idea of generating a small mass was used recently in other contexts in [44, 50 − 53] (see also [54 − 57] for similar calculations). If this Euclidean $D1$-brane (or $D3$-brane) intersects the space filling branes, which are the seven-branes in our case, in general, there are fermionic zero modes due to Ganor's strings [58] stretched between the $D1$- (or $D3$-) and the space-filling branes. These instanton zero modes will couple to the (anti)-fundamental matter fields Q and $\tilde{Q}$. Hence, upon integration of these Ganor's zero modes one can generate a non-perturbative superpotential for Q and $\tilde{Q}$. One can approach this problem by first generating a massless SQCD and then showing that one can produce the mass term by $D1$- or $D3$-brane instantons intersecting the seven-branes. It would be interesting to explore this in the future.

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A. The twist on the surface

In this appendix we will review the twisting procedure to obtain a theory on $\mathbb{R}^{3,1} \times \mathcal{S}$, where $\mathcal{S}$ is a compact Kahler surface over which we wrap the seven-branes.

We start with the maximally supersymmetric theory on $\mathbb{R}^{3,1} \times \mathbb{C}^2$. The symmetry of this theory is $\mathrm{SO}(7,1) \times \mathrm{U}(1)_R$. In addition to the eight-dimensional gauge field, this theory contains a complex scalar $\phi, \bar{\phi}$ and two fermions $\Psi_\pm$ transforming under $\mathrm{SO}(7,1) \times \mathrm{U}(1)_R$ as

$$\left(\mathbf{S}_+, \frac{1}{2}\right) \tag{A.1}$$

and

$$\left(\mathbf{S}_-, -\frac{1}{2}\right) \tag{A.2}$$

respectively. Here by $\mathbf{S}_\pm$ we denoted the positive and negative chirality representations of $\mathrm{SO}(7,1)$. This theory is invariant under two supersymmetries whose parameters $\epsilon_\pm$ transform in the same way as $\Psi_\pm$.[4] Our aim is to obtain a theory on $\mathbb{R}^{3,1} \times \mathcal{S}$ whose

[4]The simplest way to see these results is to recall that this theory can be obtained by compactifying the ten-dimensional supersymmetric Yang-Mills theory to eight dimensions.

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symmetry is reduced to $\mathrm{SO}(3,1) \times \mathrm{SO}(4) \times \mathrm{U}(1)_R$. Here $\mathrm{SO}(3,1)$ is the Lorentz group in four dimensions and $\mathrm{SO}(4)$ is the structure group of the tangent bundle of $\mathcal{S}$. The parameters $\epsilon_\pm$ decompose as follows

$$\epsilon_+ \in \left(\mathbf{S}_+, \frac{1}{2}\right) \to \left[(\mathbf{2},\mathbf{1}),(\mathbf{2},\mathbf{1}),\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}),(\mathbf{1},\mathbf{2}),\frac{1}{2}\right] \tag{A.3}$$

and

$$\epsilon_- \in \left(\mathbf{S}_-, -\frac{1}{2}\right) \to \left[(\mathbf{2},\mathbf{1}),(\mathbf{1},\mathbf{2}),-\frac{1}{2}\right] \oplus \left[(\mathbf{1},\mathbf{2}),(\mathbf{2},\mathbf{1}),-\frac{1}{2}\right], \tag{A.4}$$

where by $(\mathbf{2},\mathbf{1})$ we denote the left-handed spinor of $\mathrm{SO}(3,1)$ (or $\mathrm{SO}(4)$ depending on its position in the square brackets) and by $(\mathbf{1},\mathbf{2})$ we denote the right-handed spinor.

The twisting procedure is described by an embedding of $\mathrm{U}(1)_R$ into $\mathrm{SO}(4)$. In fact, since $\mathcal{S}$ is Kahler, its structure group is reduced to $\mathrm{U}(2)$. Thus, we need to specify how $\mathrm{U}(1)_R$ is embedded in $\mathrm{U}(2)$. It was argued in [22] that the unique choice up to isomorphism is the twist under which $\mathrm{U}(1)_R$ is embedded into the center of $\mathrm{U}(2)$. Let J be the generator of this central $\mathrm{U}(1)$. We can normalize J in such a way that under the reduction of $\mathrm{SO}(4)$ to $\mathrm{U}(2)$ the spinors of $\mathrm{SO}(4)$ transform as

$$(\mathbf{2},\mathbf{1}) \to \mathbf{2}_0, \qquad (\mathbf{1},\mathbf{2}) \to \mathbf{1}_1 \oplus \mathbf{1}_{-1}, \tag{A.5}$$

where the subscripts denote the charge under J. Then from eqs. (A.3), (A.4) and (A.5) it follows that to preserve four supercharges in four dimensions the new $\mathrm{U}(1)$ generator has to be chosen to be

$$J_{\mathrm{top}} = J \pm 2R. \tag{A.6}$$

It is easy to see that either choice of the sign leads to an isomorphic twist. We will choose $J_{\mathrm{top}} = J + 2R$. Let us check that we indeed obtain four supercharges. Under $\mathrm{SO}(3,1)\times\mathrm{U}(2)$ the supersymmetry generators transform as

$$\begin{aligned}
&\left[(\mathbf{2},\mathbf{1}),\mathbf{2}_1\right] \oplus \left[(\mathbf{1},\mathbf{2}) \otimes (\mathbf{1}_2 \oplus \mathbf{1}_0)\right], \\
&\left[(\mathbf{1},\mathbf{2}),\mathbf{2}_{-1}\right] \oplus \left[(\mathbf{2},\mathbf{1}) \otimes (\mathbf{1}_0 \oplus \mathbf{1}_{-2})\right].
\end{aligned} \tag{A.7}$$

Four-dimensional supercharges have to be scalars on $\mathcal{S}$ and, hence, correspond to the terms $(\mathbf{1},\mathbf{2}) \otimes \mathbf{1}_0$ and $(\mathbf{2},\mathbf{1}) \otimes \mathbf{1}_0$.

Now let us find how the scalars ϕ and $\bar{\phi}$ transform is the twisted theory. Before the twist they transformed as $\mathbf{1} \otimes \mathbf{1}_{\pm 1}$ under $\mathrm{SO}(3,1) \times \mathrm{U}(2)$. According to (A.6), after the twist they transform as $\mathbf{1} \otimes \mathbf{1}_{\pm 2}$. Let us interpret it geometrically. We fix conventions that the central $\mathrm{U}(1)$ of $\mathrm{U}(2)$ acts on vectors of the holomorphic vector bundle with charge $+1$. Then it acts on holomorphic differential forms with charge -1. Let $s^m, \bar{s}^{\bar{m}}$ be holomorhic and antiholomorphic coordinates on $\mathcal{S}$. Then ds^m has charge -1 and $d\bar{s}^{\bar{m}}$ has charge $+1$. Therefore, ϕ and $\bar{\phi}$ become the following differential forms

$$\phi = \phi_{mn} ds^m ds^n, \qquad \bar{\phi} = \bar{\phi}_{\bar{m}\bar{n}} \bar{s}^{\bar{m}} \bar{s}^{\bar{n}}. \tag{A.8}$$

Similarly, one can analyze the fermions. The results are summarized in subsection 2.2.

B. The twist on the curve

To discuss the theory on the intersection curve Σ we start with the untwisted theory on $\mathbb{R}^{1,1}$. This theory preserves eight supercharges and has a pair of complex scalars $(\sigma, \bar{\sigma}^c)$ forming a doublet of the R-symmetry group $\mathrm{SU}(2)_R$ and a chiral fermion (we choose its chirality to be negative) which transforms as $\mathbf{4}'$ of the Lorentz group $\mathrm{SO}(5,1)$. The supersymmetry generators transform as $\mathbf{4}' \otimes \mathbf{2}$ of $\mathrm{SO}(5,1) \times \mathrm{SU}(2)_R$. In order to twist we reduce $\mathrm{SO}(5,1)$ to $\mathrm{SO}(3,1) \times \mathrm{U}(1)$ where $\mathrm{SO}(3,1)$ is the Lorentz group in four dimensions and $\mathrm{U}(1)$ is the structure group of the tangent bundle of Σ. The representations $\mathbf{4}'$ of $\mathrm{SO}(5,1)$ decomposes under $\mathrm{SO}(3,1) \times \mathrm{U}(1)$ as

$$\mathbf{4}' \to \left[(\mathbf{2}, \mathbf{1}), -\frac{1}{2}\right] \oplus \left[(\mathbf{1}, \mathbf{2}), \frac{1}{2}\right]. \tag{B.1}$$

In addition, $\mathbf{2}$ of $\mathrm{SU}(2)_R$ decomposes to the Cartan subgroup $\mathrm{U}(1)_R$ as as

$$\mathbf{2} \to \mathbf{1}_1 \oplus \mathbf{1}_{-1}. \tag{B.2}$$

The twisting procedure is a specification of a homomorphism from $\mathrm{U}(1)_R$ to the structure group $\mathrm{U}(1)$. Let J be the generator of the structure group $\mathrm{U}(1)$. To preserve $\mathcal{N} = 1$ supersymmetry four supercharges must become scalars on Σ. This requires that the generator of the twisted $\mathrm{U}(1)$ be

$$J_{\mathrm{top}} = J \pm \frac{1}{2} R. \tag{B.3}$$

Either choice of the sign leads to an isomorphic twist. We will choose the sign to be minus.

Let us see what happens to the scalars $(\sigma, \bar{\sigma}^c)$ under this twist. Since they are scalars under $\mathrm{SO}(5,1)$ they have $J = 0$. On the other hand they carry the charge ± 1 under $\mathrm{U}(1)_R$. Thus, after the twist their charges become $\mp\frac{1}{2}$. This means that they become spinors on Σ. More precisely,

$$\sigma \in K_\Sigma^{1/2}, \qquad \bar{\sigma}^c \in \bar{K}_\Sigma^{1/2}. \tag{B.4}$$

Since the fermions do not transform under $\mathrm{SU}(2)_R$, $J_{\mathrm{top}} = J$ and the twist does not affect their geometric properties. The full spectrum is summarized in subsection 2.2. Of course, the above fields are charged under the gauge group. However, it is not affected by the twist and we have omitted the gauge group in this discussion.

C. Construction of the matrix $f_{\mathcal{C}}$

In this appendix we will present some details of construction of the 20×20 matrix $f_{\mathcal{C}}$ in subsection 4.2.

The matrix $f_{\mathcal{C}}$ provides a linear map between two 20-dimensional spaces W_1 and W_2 given by

$$\begin{aligned} W_1 &= H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma - 6F)) \simeq H^1(\sigma, \mathcal{O}(-6) \oplus \mathcal{O}(-8) \oplus \mathcal{O}(-9)), \\ W_2 &= H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma - F)) \\ &\simeq H^1(\sigma, \mathcal{O}(-1) \oplus \mathcal{O}(-3) \oplus \mathcal{O}(-4) \oplus \mathcal{O}(-5) \oplus \mathcal{O}(-6) \oplus \mathcal{O}(-7)). \end{aligned} \tag{C.1}$$

The elements $w_1 \in W_1$ and $w_2 \in W_2$ have been parametrized as follows

$$\begin{aligned} w_1 &= b_{-6}z + b_{-8}x + b_{-9}y, \\ w_2 &= c_{-3}zx + c_{-4}zy + c_{-5}x^2 + c_{-6}xy + c_{-7}y^2. \end{aligned} \tag{C.2}$$

In these expressions, b_{-i} and c_{-j} are the pullback to dP_9 of the differentials on $\sigma \simeq \mathbb{P}^1$

$$b_{-i} = \pi^* B_{-i}, \quad c_{-j} = \pi^* C_{-j}, \tag{C.3}$$

where $B_{-i} \in H^1(\sigma, \mathcal{O}(-i))$, $i = 6,\ 8,\ 9$ and $C_{-j} \in H^1(\sigma, \mathcal{O}(-j))$, $j = 3,\ 4,\ 5,\ 6,\ 7$. Furthermore, z, x, y are the variables in the Weierstrass equation

$$y^2 z = x^3 + fxz^2 + gz^3. \tag{C.4}$$

They are sections of the following line bundles on dP_9

$$z \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma)), \quad x \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma + 2F)), \quad y \in H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma + 3F)). \tag{C.5}$$

Note that each term in the sum in w_1 and w_2 in eqs. (C.2) is an element of $H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma - 6F))$ and $H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma - F))$ respectively. The map between w_1 and w_2 is given by multiplication by an element of $H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma + 5F))$ which we write as

$$\mathcal{C} = a_5 z + a_3 x + a_2 y, \tag{C.6}$$

where $a_k \in H^0(dP_9, \pi^* \mathcal{O}(k))$, $k = 1,\ 3,\ 5$. This means that $a_k = \pi^* A_k$, where A_k is a homogeneous polynomial of degree k on σ. In subsection 4.2 we introduce homogeneous coordinates (u, v) on σ and parametrized A_k as follows

$$\begin{aligned} A_5 &= \psi_1 u^5 + \psi_2 u^4 v + \psi_3 u^3 v^2 + \psi_4 u^2 v^3 + \psi_5 u v^4 + \psi_6 v^5, \\ A_3 &= \phi_1 u^3 + \phi_2 u^2 v + \phi_3 u v^2 + \phi_4 v^3, \\ A_2 &= \chi_1 u^2 + \chi_2 uv + \chi_3 v^3, \end{aligned} \tag{C.7}$$

where $\{\psi_a, \phi_b, \chi_c\}$ are the *projective* vector bundle moduli. To simplify our notation, we will remove the pullback symbol π^* and identify $b_{-i} = B_{-i}$, $c_{-j} = C_{-j}$ and $a_{-k} = A_{-k}$ and view the coefficients b_{-i}, c_{-j} and a_k in eqs. (C.2) and (C.6) as differentials and polynomials on σ.

Suppressing for the time being the coefficients b_{-i} and c_{-j} we see that W_1 us spanned by the the following basis blocks

$$z, x, y. \tag{C.8}$$

Similarly, W_2 is spanned by the basis blocks

$$zx, xy, x^2, xy, y^2. \tag{C.9}$$

Now we multiply w_1 in eq. (C.2) by $\mathcal{C}$ in eq. (C.6) and expand the answer in basis elements in (C.9). We obtain the following matrix $f_{\mathcal{C}}$

$$f_{\mathcal{C}} = \begin{array}{c} \\ xz \\ yz \\ x^2 \\ xy \\ y^2 \end{array} \begin{array}{c} \begin{array}{ccc} z & x & y \end{array} \\ \begin{pmatrix} a_3 & a_5 & 0 \\ a_2 & 0 & a_5 \\ 0 & a_3 & 0 \\ 0 & a_2 & a_3 \\ 0 & 0 & a_2 \end{pmatrix} \end{array}. \tag{C.10}$$

The matrix $f_{\mathcal{C}}$ is written in a block form where each block is a $(j-1) \times (i-1)$ matrix for $j = 3, 4, 5, 6, 7$ and $i = 6, 8, 9$. Now we can compute each block by expanding a_k in the coordinates (u, v) as in (C.7). For example, let us compute the $z - zx$ block a_3. That is, we want to compute the map

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma - 6F))|_{b_{-6}} \xrightarrow{a_3} H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma - F))|_{c_{-3}}. \tag{C.11}$$

The map is a multiplication by a_3 in (C.7). Note that

$$h^1(dP_9, \mathcal{O}_{dP_9}(3\sigma - 6F))|_{b_{-6}} = h^1(\sigma, \mathcal{O}(-6)) = 5 \tag{C.12}$$

and

$$h^1(dP_9, \mathcal{O}_{dP_9}(6\sigma - F))|_{c_{-3}} = h^1(\sigma, \mathcal{O}(-3)) = 2. \tag{C.13}$$

Therefore, the block $z - zx$ in the matrix (C.10) is a 2×5 matrix. To construct a_3 in (C.11) and (C.10) we use the Serre duality to identify

$$H^1(\sigma, \mathcal{O}(-6) = H^0(\sigma, \mathcal{O}(4))^\vee \tag{C.14}$$

and

$$H^1(\sigma, \mathcal{O}(-3) = H^0(\sigma, \mathcal{O}(1))^\vee \tag{C.15}$$

Let us introduce the two-dimensional linear space

$$\hat{V} = H^0(\sigma, \mathcal{O}(1)). \tag{C.16}$$

It is parametrized by the linear functions on σ. That is, by the projective coordinates (u, v). Similarly, we introduce the dual vector space

$$\hat{V}^\vee = H^0(\sigma, \mathcal{O}(1))^\vee \tag{C.17}$$

and parametrize it by the dual basis (u^*, v^*), where

$$u^* u = 1, \quad v^* v = 1, \quad u^* v = u v^* = 0. \tag{C.18}$$

Then from eq. (C.14) it follows that $H^1(\sigma, \mathcal{O}(-6))$ is spanned by the following basis

$$\{u^{*4}, u^{*3}v^*, u^{*2}v^{*2}, u^* v^{*3}, v^{*4}\}. \tag{C.19}$$

Similarly, $H^1(\sigma, \mathcal{O}(-3)) \simeq \hat{V}^\vee$ is spanned by

$$\{u^*, v^*\}. \tag{C.20}$$

The coefficient a_3 is a map between (C.19) and (C.20). Multiplying basis elements in (C.19) by a_3 in (C.7) and using relations (C.18) we obtain the following 2×5 matrix

$$\begin{array}{c} \\ u^* \\ v^* \end{array} \begin{array}{c} \begin{array}{ccccc} u^{*4} & u^{*3}v^* & u^{*2}v^{*2} & u^* v^{*3} & v^{*4} \end{array} \\ \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 \\ 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{pmatrix} \end{array}. \tag{C.21}$$

Continuing this way, one can build up the complete matrix $f_{\mathcal{C}}$. The determinant of this matrix is given in eq. (4.49).

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